Vortex line representation for flows of ideal and viscous fluids [1]

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Abstract

It is shown that the Euler hydrodynamics for vortical flows of an ideal fluid coincides with the equations of motion of a charged *compressible* fluid moving due to a self-consistent electromagnetic field. Transition to the Lagrangian description in a new hydrodynamics is equivalent for the original Euler equations to the mixed Lagrangian-Eulerian description - the vortex line representation (VLR) [2]. Due to compressibility of a "new" fluid the collapse of vortex lines can happen as the result of breaking (or overturning) of vortex lines. It is found that the Navier-Stokes equation in the vortex line representation can be reduced to the equation of the diffusive type for the Cauchy invariant with the diffusion tensor given by the metric of the VLR.

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1. Collapse as a process of a singularity formation in a finite time from the initially smooth distribution plays the very important role being considered as one of the most effective mechanisms of the energy dissipation. For hydrodynamics of incompressible fluids collapse must play also a very essential role. It is well known that appearance of singularity in gasodynamics, i.e., in compressible hydrodynamics, is connected with the phenomenon of breaking that is the physical mechanism leading to emergence of shocks. From the point of view of the classical catastrophe theory [3] this process is nothing more than the formation of folds. It is completely characterized by the mapping corresponding to transition from the Eulerian description to the Lagrangian one. Vanishing the Jacobian J of this mapping means emergence of a singularity for spatial derivatives of velocity and density of a gas. In the incompressible case breaking as intersection of trajectories of Lagrangian particles is absent because the Jacobian of the corresponding mapping is fixed, in the simplest case equal to unity. By this reason, it would seem that there were

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no any reasons for existence of such phenomenon at all. In spite of this fact, as it was shown in [4, 5, 6], breaking, however, is possible in this case also. It can happen with vortex lines. Unlike the breaking in gasodynamics, the breaking of vortex lines means that one vortex line reaches another vortex line. For smooth initial conditions breaking happens first time while touching vortex lines at a single point. In the touching point the vorticity becomes infinite. And this is possible in spite of incompressibility of both divergence-free fields, i.e., vorticity and velocity. To describe the breaking of vortex lines in the papers [2, 7] it was suggested the vortex line representation – a mixed Lagrangian-Eulerian description when each vortex line is labeled by a two-dimensional marker and another parameter defines the vortex line itself.

This paper is devoted to development of this method to apply to both ideal and viscous fluids. We clarify the role of the Clebsch variables in the vortex line representation: these variables can be used as Lagrangian markers of vortex lines. However, as well known, these variables can be introduced always only locally and, generally speaking, can not be extended for the whole space. In the general situation we demonstrate in this paper that transition to the vortex line representation is equivalent to consideration of a new *compressible* hydrodynamics of a charged fluid flowing under action of a self-consistent electromagnetic field. In this case the electric and magnetic fields satisfy the Maxwell equations. The most essential property of a new hydrodynamics is a compressibility of a new fluid that for its Lagrangian description means compressibility of the corresponding mapping and, respectively, a possibility of a breaking. In terms of the Eulerian characteristics this results in the breaking of vortex lines when the vorticity $\mathbf{\Omega} = \text{curl } \mathbf{v}$ takes infinite value. In the framework of the new hydrodynamics of a charged fluid the role of density plays the quantity inverse to J which is naturally called as a density of vortex lines. This quantity appears from the Cauchy formula for the vorticity $\mathbf{\Omega}$. Evolution of the vortex line density in time and space is defined by the velocity component normal to the vorticity. As it is shown in this paper the Cauchy formula can be obtained from a "new" Kelvin theorem as well as from the analog of the Weber transformation. As the result, the Euler equations turn out to be resolved with respect to the Cauchy invariants, i.e., relative to the infinite number of integrals of motion. In this case one can consider the Euler equations as the partially integrated equations. This circumstance is very important for numerical solution of the Euler equation.

The vortex line representation can be applied not only to ideal hydrodynamics but also to flow description of viscous incompressible fluids in the framework of the Navier-Stokes equation. In the paper we obtain the equation of the diffusion type describing dynamics of the Cauchy invariant in the viscous case with the "diffusion tensor" determined by the VLR metric. In its form this equation coincides with the equation derived in [8]. In this case the equations of motion of vortex lines in its original (for ideal fluids) form are understood as the equations given the transformation to a new curvilinear system of coordinates. The obtained exact equations for description of viscous flows can be considered as the result of exact separation of two different temporal scales: the inertial (in fact, nonlinear) scale and the viscous one.

2. As well known (see, for instance, [9], [10]) the Euler equations for an ideal incompressible fluid,

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\nabla)\mathbf{v} = -\nabla p, \qquad \text{div } \mathbf{v} = 0, \tag{1}$$

in both two-dimensional and three-dimensional cases possess the infinite (continuous) number of integrals of motion. These are the so called Cauchy invariants. The most simple way to derive the Cauchy invariants is one to use the Kelvin theorem about conservation of the velocity circulation,

$$\Gamma = \oint (\mathbf{v} \cdot d\mathbf{l}), \tag{2}$$

where the integration contour $C[\mathbf{r}(t)]$ moves together with a fluid. If in this expression one makes a transform from the Eulerian coordinate $\mathbf{r}$ to the Lagrangian ones $\mathbf{a}$ then Eq. (2) can be rewritten as follows:

$$\Gamma = \oint \dot{x}_i \cdot \frac{\partial x_i}{\partial a_k}\, da_k \ ,$$

where a new contour $C[\mathbf{a}]$ is already immovable. Hence, due to arbitrariness of the contour $C[\mathbf{a}]$ and using the Stokes formula one can conclude that the quantity

$$\mathbf{I} = \text{rot}_a \left(\dot{x}_i \frac{\partial x_i}{\partial \mathbf{a}} \right) \tag{3}$$

conserves in time at each point $\mathbf{a}$. This is just the Cauchy invariant. If the Lagrangian coordinates $\mathbf{a}$ in (3) coincide with the initial positions of fluid particles the invariant $\mathbf{I}$ is equal to the initial vorticity $\mathbf{\Omega}_0(\mathbf{a})$.

Conservation of these invariants, as it was shown first by Salmon [10], is consequence of the special (infinite) symmetry - the so-called relabeling symmetry. The Cauchy invariants characterize the frozenness of the vorticity into fluid. This is a very important property according to which fluid (Lagrangian) particles can not leave its own vortex line where they were initially. Thus, the Lagrangian particles have one independent degree of freedom – motion along vortex line. From another side, such a motion as it follows from the equation for the vorticity

$$\frac{\partial \mathbf{\Omega}}{\partial t} = \text{rot } [\mathbf{v} \times \mathbf{\Omega}], \tag{4}$$

does not change its value. From this point of view a vortex line represents the invariant object and therefore it is natural to seek for such a transformation when this invariance is seen from the very beginning. Such type of description - the vortex line representation - was introduced in the papers [2, 7] by Ruban and the author of this paper.

3. Consider the vortical flow ($\mathbf{\Omega} \neq 0$) of an ideal fluid given by the Clebsch variables λ and μ:

$$\mathbf{\Omega} = [\nabla\lambda \times \nabla\mu]. \tag{5}$$

The geometrical meaning of these variables is well known: intersection of two surfaces λ = const and μ = const yields the vortex line. It is known also that in the incompressible case the Clebsch variables are Lagrangian invariants, being unchanged along trajectories of fluid particles:

$$\frac{\partial\lambda}{\partial t} + (\mathbf{v}\nabla)\lambda = 0; \qquad \frac{\partial\mu}{\partial t} + (\mathbf{v}\nabla)\mu = 0. \tag{6}$$

Therefore these variables can be taken as markers for vortex lines. It is easily to establish that transition in (5) to new variables

$$\lambda = \lambda(x,y,z), \qquad \mu = \mu(x,y,z), \qquad s = s(x,y,z), \tag{7}$$

where s is the parameter given the vortex line, leads to the expression

$$\mathbf{\Omega}(\mathbf{r},t) = \frac{1}{J}\cdot\frac{\partial\mathbf{R}}{\partial s} \tag{8}$$

where

$$J = \frac{\partial(x,y,z)}{\partial(\lambda,\mu,s)} \tag{9}$$

is the Jacobian of the mapping

$$\mathbf{r} = \mathbf{R}(\lambda,\mu,s). \tag{10}$$

The transform (10) inverse to (7) defines the corresponding transition to the curvilinear, connected with vortex lines, system of coordinates.

The equations of motion of vortex lines - the equations for $\mathbf{R}(\lambda,\mu,s,t)$ – can be obtained directly from the equation of motion for the vorticity (4). However, the most simple way to derive them is to use the combination of the equations (6):

$$\nabla\mu\left[\frac{\partial\lambda}{\partial t} + (\mathbf{v}\nabla)\lambda\right] - \nabla\lambda\left[\frac{\partial\mu}{\partial t} + (\mathbf{v}\nabla)\mu\right] = 0, \tag{11}$$

which is identical to (6) due to a linear independence of the vectors $\nabla\lambda$ and $\nabla\mu$.

Performing in (11) the transformations (7), we arrive at the equation of motion for vortex lines [2]:

$$\left[\frac{\partial\mathbf{R}}{\partial s} \times \left(\frac{\partial\mathbf{R}}{\partial t} - \mathbf{v}(\mathbf{R},t)\right)\right] = 0. \tag{12}$$

This equation has one important property: any motion along a vortex line does not change the line itself. It is easily to check that Eq. (12) is equivalent to the equation

$$\frac{\partial \mathbf{R}}{\partial t} = \mathbf{v}_n(\mathbf{R}, t), \tag{13}$$

where $\mathbf{v}_n$ is the velocity component normal to the vorticity vector.

In accordance with the Darboux theorem, the Clebsch variables can be introduced locally always but not globally. It is well known also that the flows parameterized by the Clebsch variables has a zero helicity integral $\int (\mathbf{v} \cdot \text{rot } \mathbf{v}) d\mathbf{r}$ – the topological invariant which characterizes a degree of knottiness of vortex lines. Therefore to introduce the vortex line representation for flows with nontrivial topology it is necessary to come back to the original equations of motion (1) and (4) for velocity and vorticity.

4. According to the equation (4) the tangent to the vector $\mathbf{\Omega}$ velocity component $\mathbf{v}_\tau$ does not effect (directly) on the vorticity dynamics, i.e., in (4) we can put, instead of $\mathbf{v}$, its transverse component $\mathbf{v}_n$.

The equation of motion for the transverse velocity $\mathbf{v}_n$ follows directly from the equation (1). It has the form of the equation of motion of charged particle moving in an electromagnetic field:

$$\frac{\partial \mathbf{v}_n}{\partial t} + (\mathbf{v}_n \nabla)\mathbf{v}_n = \mathbf{E} + [\mathbf{v}_n \times \mathbf{H}], \tag{14}$$

where the effective electric and magnetic fields are given by the expressions:

$$\mathbf{E} = -\nabla \left(p + \frac{v_\tau^2}{2} \right) - \frac{\partial \mathbf{v}_\tau}{\partial t}, \tag{15}$$

$$\mathbf{H} = \text{rot } \mathbf{v}_\tau. \tag{16}$$

Interesting to note that the electric and magnetic fields introduced above are expressed through the scalar φ and vector $\mathbf{A}$ potentials by the standard way:

$$\varphi = p + \frac{\mathbf{v}_\tau^2}{2}, \qquad \mathbf{A} = \mathbf{v}_\tau, \tag{17}$$

so that two Maxwell equations

$$\text{div } \mathbf{H} = 0, \qquad \frac{\partial \mathbf{H}}{\partial t} = -\text{rot } \mathbf{E}$$

satisfy automatically. In this case the vector potential $\mathbf{A}$ has the gauge

$$\text{div } \mathbf{A} = -\text{div } \mathbf{v}_n,$$

which is equivalent to the condition div $\mathbf{v} = 0$.

Two other Maxwell equations can be written also but they can be considered as definition of the charge density ρ and the current $\mathbf{j}$ which follow from the relations (15) and (16). The basic equation in the new hydrodynamics is the equation of motion (14) for the normal component of the velocity which represents the equation of motion for nonrelativistic particle with a charge and a mass equal to unity, the light velocity in this units is equal to 1.

The equation of motion (14) is written in the Eulerian representation. To transfer to its Lagrangian formulation one needs to consider the equations for "trajectories" given by the velocity $\mathbf{v}_n$:

$$\frac{d\mathbf{R}}{dt} = \mathbf{v}_n(\mathbf{R}, t) \tag{18}$$

with initial conditions

$$\mathbf{R}|_{t=0} = \mathbf{a}.$$

Solution of the equation (18) yields the mapping

$$\mathbf{r} = \mathbf{R}(\mathbf{a}, t), \tag{19}$$

which defines transition from the Eulerian description to a new Lagrangian one.

The equations of motion in new variables are the Hamilton equations:

$$\dot{\mathbf{P}} = -\frac{\partial h}{\partial \mathbf{R}}, \qquad \dot{\mathbf{R}} = \frac{\partial h}{\partial \mathbf{P}}, \tag{20}$$

where dot means differentiation with respect to time for fixed $\mathbf{a}$, $\mathbf{P} = \mathbf{v}_n + \mathbf{A} \equiv \mathbf{v}$ is the generalized momentum, and the Hamiltonian of a particle h being a function of momentum $\mathbf{P}$ and coordinate $\mathbf{R}$ is given by the standard expression:

$$h = \frac{1}{2}(\mathbf{P} - \mathbf{A})^2 + \varphi \equiv p + \frac{\mathbf{v}^2}{2},$$

i.e., coincides with the Bernoulli "invariant".

The first equation of the system (20) is the equation of motion (14), written in terms of $\mathbf{a}$ and t, and the second equation coincides with (18).

For new hydrodynamics (14) or for its Hamilton version (20) it is possible to formulate a "new" Kelvin theorem (it is also the Liouville theorem):

$$\Gamma = \oint (\mathbf{P} \cdot d\mathbf{R}), \tag{21}$$

where integration is taken along a loop moving together with the "fluid". Hence, analogously as it was made before while derivation of (3) we get the expression for a new Cauchy invariant:

$$\mathbf{I} = \mathrm{rot}_a \left(P_i \frac{\partial x_i}{\partial \mathbf{a}} \right). \tag{22}$$

Its difference from the original Cauchy invariant (3) consists in that in the equation of motion (18) instead of the velocity $\mathbf{v}$ stands its normal component $\mathbf{v}_n$. As consequence, the "new" hydrodynamics becomes compressible: div $\mathbf{v}_n \neq 0$. Therefore on the Jacobian J of the mapping (19) there are imposed no restrictions. The Jacobian J can take arbitrary values.

From the formula (22) it is easily to get the expression for the vorticity $\mathbf{\Omega}$ in the given point $\mathbf{r}$ at the instant t (compare with [2, 7]):

$$\mathbf{\Omega}(\mathbf{r}, t) = \frac{(\mathbf{\Omega}(\mathbf{a}) \cdot \nabla_a)\mathbf{R}(a, t)}{J}, \tag{23}$$

where J is the Jacobian of the mapping (19) equal to

$$J = \frac{\partial(x_1, x_2, x_3)}{\partial(a_1, a_2, a_3)}.$$

Here we took into account that the generalized momentum $\mathbf{P}$ coincides with the velocity $\mathbf{v}$, including the moment of time $t = 0$: $\mathbf{P}_0(\mathbf{a}) \equiv \mathbf{v}_0(\mathbf{a})$. $\mathbf{\Omega}_0(\mathbf{a})$ in this relation is the "new" Cauchy invariant with zero divergence: $\mathrm{div}_a \mathbf{\Omega}_0(a) = 0$.

The representation (23) generalizes the relation (5) to an arbitrary topology of vortex lines. The variables $\mathbf{a}$ in this expression can be considered locally as a set of λ, μ and s.

As known (see, for instance, [7]), expression for the Cauchy invariant can be obtained from the Weber transformation. This is the representation of velocity in terms of the initial data which can be obtained by integrating the Cauchy invariant (23).

Consider the following one-form $\omega = (\mathbf{P} \cdot d\mathbf{R})$ and calculate its time derivative. By means of the equations of motion (20) we get:

$$\dot{\omega} = d[-h + (\mathbf{P}\dot{\mathbf{R}})].$$

Hence it follows that the vector function

$$u_k = \frac{\partial x_i}{\partial a_k} \cdot P_i,$$

dependent on t and $\mathbf{a}$, will obey the following equation of motion:

$$\dot{u}_k = \frac{\partial}{\partial a_k}\left(-p + \frac{v_n^2}{2} - \frac{v_\tau^2}{2}\right).$$

Integration of this equation in time gives the Weber-type transformation:

$$u_k(\mathbf{a}, t) = u_{k0}(\mathbf{a}) + \frac{\partial \Phi}{\partial a_k}, \tag{24}$$

where the potential Φ satisfies the nonstationary Bernoulli equation:

$$\dot{\Phi} = -p + \frac{v_n^2}{2} - \frac{v_\tau^2}{2}.$$

If $\Phi|_{t=0} = 0$ then the time independent vector $\mathbf{u}_0(\mathbf{a})$ coincides with the initial velocity $\mathbf{v}_0(\mathbf{a})$. By applying the operator curl to the relation (24) we arrive again at the Cauchy invariant (22).

Thus, in the general situation the equation of motion of vortex lines has the form (18) which is completed by the relation (23) and the equation

$$\mathbf{\Omega}(\mathbf{r}, t) = \mathrm{rot}_r \mathbf{v}(\mathbf{r}, t) \tag{25}$$

with additional constraint $\mathrm{div}_r \mathbf{v}(\mathbf{r}, t) = 0$.

The equations of motion (18), (25) together with the relation (23) can be considered as the result of partial integration of the Euler equation (1). These new equations are resolved with respect to the Cauchy invariants – an infinite number of integrals of motion, that is a very important issue for numerical integration (see [5, 6]). For the partially integrated system the Cauchy invariants conserve automatically that, however, for direct numerical integration of the Euler equation one needs to test in which extent these invariants remain constant. Probably, this is one of the main restrictions defining accuracy of discrete algorithms of direct integration of the Euler equations.

Another very important property of the vortex line representation is absence of any restrictions on the value of the Jacobian J which do exist, for instance, for transition from the Eulerian description to the Lagrangian one in the original Euler equation (1) when Jacobian in the simplest situation is equal to unity. The value $1/J$ for the system (18), (25), (23) has a meaning of a density n of vortex lines. This quantity as a function of $\mathbf{r}$ and t , according to (18, obeys the discontinuity equation:

$$\frac{\partial n}{\partial t} + \mathrm{div}_r(n\mathbf{v}_n) = 0. \tag{26}$$

In this equation $\mathrm{div}_r \mathbf{v}_n \neq 0$ because only the total velocity has zero divergence.

5. Consider now the question about application of the VLR to flows of viscous fluids. Write down the Navier-Stokes equation for vorticity $\mathbf{\Omega}$:

$$\frac{\partial \mathbf{\Omega}}{\partial t} = \mathrm{rot}[\mathbf{v} \times \mathbf{\Omega}] - \nu \ \mathrm{rot}\ \mathrm{rot}\ \mathbf{\Omega}, \tag{27}$$

and perform in this equation the transform to new variables $\mathbf{a}$ and t by means of changes defined by the equation (18) together with the Cauchy relation (23) where $\mathbf{\Omega}_0$ is assumed a function of not only $\mathbf{a}$ but also time t: $\mathbf{\Omega}_0 = \mathbf{\Omega}_0(\mathbf{a}, t)$.

Then after substitution (23) into (27) the first term in the right hand side is cancelled because of (18). At the result, the equation (27) is written in the form:

$$\frac{1}{J}\left(\frac{\partial \mathbf{\Omega}_0}{\partial t} \cdot \nabla_a\right)\mathbf{R} = -\nu \ \mathrm{rot}\ \mathrm{rot}\left\{\frac{1}{J}(\mathbf{\Omega}_0 \cdot \nabla_a)\mathbf{R}\right\}. \tag{28}$$

Next, change differentiation relative to $\mathbf{r}$ in the r.h.s. of (28) to differentiation against $\mathbf{a}$. After simple, but cumbersome calculations the equation (28) transforms into the equation for $\mathbf{\Omega}_0(\mathbf{a}, t)$:

$$\frac{\partial \mathbf{\Omega}_0}{\partial t} = -\nu \ \mathrm{rot}_a \left(\frac{\hat{g}}{J} \mathrm{rot}_a \left(\frac{\hat{g}}{J} \mathbf{\Omega_0} \right) \right). \tag{29}$$

Formally it is a linear equation for $\mathbf{\Omega}_0$, here $\hat{g}$ is the metric tensor equal to

$$g_{\alpha\beta} = \frac{\partial x_i}{\partial a_\alpha} \cdot \frac{\partial x_i}{\partial a_\beta}.$$

The equation (29) for the Cauchy invariant formaly coincides with that obtained by Zenkovich and Yakubovich for incompressible hydrodynamics [8] in which the variables $\mathbf{a}$ are assumed to be Lagrangian markers of fluid particles. In the Zenkovich-Yakubovich equation the Jacobian J is proposed to be independent on time, in the simplest case equal to 1. Just this is a principle difference between the Zenkovich-Yakubovich equation and the equation (29). J in (29) is a function of time t and coordinates $\mathbf{a}$.

Remarkable peculiarity of the obtained system is the *exact separation* of two different temporal scales, responsible for the inertial (in fact, nonlinear) processes and for the viscous processes. The former ones are described by the equation (18), and the latter by the equation of diffusive type (29) in which the diffusion "coefficient", proportional to viscosity ν, is defined by the metric of the mapping $\mathbf{r} = \mathbf{R}(\mathbf{a}, t)$.

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References

[1] E.A. Kuznetsov, Pis'ma v ZhETF (JETP Letters), **76**, no 6 (2002).

[2] E.A. Kuznetsov, V.P. Ruban, Pis'ma v ZhETF, **67**, 1015 (1998) [JETP Letters **67**, 1076 (1998)].

[3] V.I. Arnold, *Theory of Catastrophe*, Znanie, Moscow , 1981 (in Russian) [English transl.: *Theory of Catastrophe,* 1986, 2nd rev. ed. Springer].

[4] E.A. Kuznetsov, V.P. Ruban, ZhETF **118**, 893 (2000) [JETP **91**, 776 (2000)].

[5] V.A. Zheligovsky, E.A. Kuznetsov, and O.M.Podvigina, Pis'ma v ZhETF, **74**, 402 (2001) [JETP Letters **74**, 367 (2001)].

[6] E.A. Kuznetsov, O.N.Podvigina and V.A. Zheligovsky, *Numerical evidence of breaking of vortex lines in an ideal fluid*, in: Proceedings of IUTAH Symp. "Tubes, Sheets and Singularities in Fluid Dynamics", Zakopane, Kluver (2002) (in press).

[7] E.A. Kuznetsov and V.P.Ruban, Phys. Rev. E, **61**, 831 (2000).

[8] E.I. Yakubovich, D.A. Zenkovich, *Matrix Fluid Dynamics*, Proc. of Int. Conf. "Progress in Nonl. Science" (July 2001, N.Novgorod, Russia), vol. II "Frontiers of Nonlinear Physics", ed. by A.G. Litvak, Nizhny Novgorod, 2002, pp. 282-287; physics/0110004.

[9] V.E. Zakharov and E.A. Kuznetsov, Physics Uspekhi **137**, 1137 (1997) (in Russian).

[10] R. Salmon, Ann. Rev. Fluid Mech., **20**, 225 (1988).